Efficiency in Simulating Information Networks

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ABSTRACT

Recent successes in quantum technology have provided a number of powerful tools for viewing nature in terms of information. In particular, quantum information science has proved to be a useful way to greatly simplify the way the universe is viewed as well as treating the subject, an observer, and the object, the observed universe, on an equal footing. Several important applications in quantum technology such as communication, information processing, etc., use a distinctive property of entanglement. In this article, a numerical simulation is applied to a swapping measurement protocol involving two 3-level entangled states. A different condition, in non-maximal states, that approximates the optimal result (i.e., the weaker link between the two initial states) will be shown through a numerical method. This different class of states is distributed just as wide as the class yielding the optimal result. The result may be useful in establishing a long distance maximal entanglement that is often fragile and unstable due to noise.

Key Words: Quantum Information Science, Numerical Methods, Entanglement, Nonlocality

Introduction

The ancient Greek philosopher Plato believed in the existence of an ideal world separate from our imperfect world. It is interesting to note that while our physical world is discrete and imperfect, mathematical modeling of nature is often done by using an ideal concept such as a perfect sphere. This is rather surprising when the observer is thought to be part of an imperfect physical system yet is believed able to imagine, or generate, ideal concepts.

When the founding fathers of quantum mechanics first outlined their theory, there was confusion and debate over its exact nature that continue to this day. Indeed, the necessity of placing the observer on an equal footing with the object being observed was not easily accepted. For example, the unexpected subjectivity involved in quantum theory may be understood through John Bell’s comments on the two titans clashing over the true nature of quantum theory (Farmelo, 2010):

“Bohr was inconsistent, unclear, willfully obscure and right. Einstein was consistent, clear, down-to-earth and wrong.”

Indeed, the subjective nature of quantum theory does not occur to be consistent with the earlier practice in science where an objective reality was generally pursued.

The philosopher of science Karl Popper discussed the fundamental limitations of a common practice used in science, namely, inductive reasoning, which attempts to provide a general rule based on limited examples. For instance, one may not conclude “all swans are white” based on observing a finite number of swans. No matter how many swans have been observed to be white, there is always a possibility that the next swan will not be white. Popper implied the subjectivity involved in making observations as follows (Popper, 2002):

“Observation is always selective. It needs a chosen object, a definite task, an interest, a point of
view, a problem.”

Maximal Entanglement

Entanglement has been shown to be an important asset in several applications in information technology including communication (Cleve et al., 1997), key distribution (Ekert, 1991) and information processing (Ladd et al., 2010; Kok et al., 2007). These powerful applications often use a special type of entanglement known as the maximally correlated states. There have been various studies conducted turning non-maximal entanglements into maximal ones with local operations and classical communications (Bennett et al., 1996; Jonathan et al., 1999; Hardy, 1999; Shi et al., 2000; Hardy et al., 2000; Lo et al., 2001; Hsu, 2002; Modlawska et al., 2008; Yang et al., 2009). In (Jonathan et al., 1999; Hardy, 1999), given an entangled state with ordered Schmidt coefficients,

$$|\psi\rangle = \sum_{j=0}^{n} \sqrt{\gamma_j} |j\rangle_{n}$$  \hspace{1cm} (1)

the maximum average entanglement was shown to yield

$$E_{\text{max}} = \sum_{j=0}^{n} (\gamma_{j-1} - \gamma_j) / \log_2 l$$  \hspace{1cm} (2)

where $\gamma_n = 0$. Another important aspect of the practical application of entanglement is regarding distance. There have been various efforts made to create a long-distance entanglement to examine this spooky action at a distance (Briegel et al., 1998; Dür et al., 1999; Waks et al., 2002). In 1998, a research team was successful in creating an entangled pair that was ~10km apart (Tittel et al., 1998). More recently, much longer entanglements were experimentally generated, around 100km in 2012 (Yin et al., 2012; Ma et al., 2012) and ~1,200km in 2017 (Yin et al., 2017).

While there have been some remarkable achievements in establishing long distance quantum entanglement, it is still helpful to have more techniques to create and manipulate long distance correlation. One such method is entanglement swapping (Zukowski et al., 1993) (Fig. 1), which connects multiple short entanglements to create a long one, by performing measurements at each joint (also see (Pan et al., 1998; Jennewein et al., 2001; Sciarrino et al., 2002; Boulant et al., 2003; Riebe et al., 2008; Kaltenbaek et al., 2009) for experimental realizations). Since entanglement swapping protocols initially relied on maximally entangled states, a question arose asking if there could be longer entanglements generated from non-maximal initial states. It was shown that, for 2-level entangled states, Bell measurement yields the optimal result by creating entanglement of the weaker link between the two (Bose et al., 1998; Shi et al., 2000).

Two 3-Level States

Let us consider the following two entangled states:

$$|\phi\rangle_{12} = \sum_{i=0}^{3} \sqrt{\alpha_i} |i\rangle_{1}$$  \hspace{1cm} (3)

$$|\psi\rangle_{23} = \sum_{j=0}^{3} \sqrt{\beta_j} |j\rangle_{2}$$  \hspace{1cm} (4)

where we assume the coefficients are ordered. The Bell measurement can be made on particles 2 and 3 with the following basis:

$$|\Phi^\nu_{\omega}\rangle_{23} = \frac{1}{\sqrt{3}} \sum_{k=0}^{2} e^{i \nu (k+2\omega)/3} |k, k + \omega\rangle$$  \hspace{1cm} (5)

where $\nu, \omega = 0,1,2$. It may be checked to determine that these 9 states are orthogonal. The Bell measurements on particles 2 and 3 yield the long entangled states between 1 and 4 with the following coefficients with cancellation of phase terms:

$$\alpha_0 \beta_0, \alpha_2 \beta_2, \alpha_3 \beta_1$$

$$\alpha_1 \beta_2, \alpha_2 \beta_3, \alpha_3 \beta_0$$  \hspace{1cm} (6)

The coefficients in the $\mu$th row ($\mu = 0,1,2$) in (6) are assumed to be ordered. It may be considered that the largest value corresponds to $L^\omega_{\nu}$, the next one to $L^\nu\omega$ and the last to $L^\nu\nu$. The average maximal entanglement may be calculated using (2) as

$$\sum_{\nu=1}^{2} (L^\nu\nu - L^\nu_{\omega}) \nu \log_2 \nu$$

where $L^\nu_{\nu} = 0$.

Numerical Analysis

Unlike 2-level states, the swapping measurement of 3-level entangled states does not
always yield the weaker link. For instance, let us consider the case where
\[ \alpha_i = \beta_i, \quad i = 0,1,2 \quad (7) \]
that is, two equal 3-level states. As shown in Fig. 2, the average entanglement is generally lower than the initial entanglement of either one of the two states. To consider the case in which the weaker link is achievable, the following conditions may be considered (Hardy et al., 2000):

\[ (c1) \quad \alpha_i \beta_i \geq \alpha_2 \beta_2 \geq \alpha_3 \beta_3 \quad (8) \]
\[ (c2) \quad \alpha_1 \beta_2 \geq \alpha_2 \beta_3 \geq \alpha_3 \beta_1 \quad (9) \]
\[ (c3) \quad \alpha_1 \beta_3 \geq \alpha_2 \beta_1 \geq \alpha_3 \beta_2 \quad (10) \]

In (Hardy, 1999), a simple way of manipulating entanglement has been discussed using an area diagram. As shown in Fig. 3, each coefficient of the outcome may be represented through a box with a unit width and height corresponding to the coefficients. As seen in (2), (ii) of Fig. 3 may be used to calculate the maximum entanglement by adding the area of each box multiplied by \( \log_2 \). When conditions (8-10) are met, the coefficients may first be added up as seen in (i) of Fig. 3, which then only leaves the coefficients of \( \alpha_i \)’s. This is the weaker link between the two and is therefore the optimal value.

Although it is good to have certain non-maximal states that generate a longer entanglement with a weaker link, it is still desirable to have other coefficients that yield an outcome close to the optimal case. Let us now consider the case where the conditions in (8-10) are not necessarily true; that is, when (c2) is replaced by

\[ (\alpha_1 \beta_1, \alpha_2 \beta_2, \alpha_3 \beta_3) \]

![Figure 2](image.png)

**Figure 2.** For two equal 3-level entangled states, Bell measurement does not generally yield the weaker link, which is either one of two initial states. This figure represents the weaker link (straight line) compared with the average maximal entanglement (dashed line) using \( \alpha_i = \alpha_1 \) and \( 0.3 \leq \alpha_i \leq 1 \).

![Figure 3](image.png)

**Figure 3.** An area diagram helps to visualize the process of calculating maximal entanglement, when the coefficients are particularly ordered. As seen in (ii), the entanglement calculated from each state added up is equal to the weaker link as seen in (i) (i.e., only with \( \alpha_i \)’s).
\((c'2)\ \alpha_i \beta_j > \alpha_k \beta_j > \alpha_i \beta_k\) (11)

In Fig. 4, the coefficients of \(\alpha_i\)'s and \(\beta_i\)'s that satisfy conditions (c1), (c2') and (c3), while approximating the optimal entanglement (i.e., the weaker link) with < 0.01 margin, is shown. The top of Fig. 4 shows the coefficients \(\alpha_i\)'s in (3) while the bottom of the figure displays non-maximal states in (4), which resembles the weaker link. Therefore, the numerical approach shows a different class of non-maximal states satisfying (c1), (c2') and (c3) that approximates the optimal entanglement previously considered with conditions (8-10). Not only are these states an approximate of the optimal outcome, but they are distributed just as widely as the class of non-maximal states yielding the optimal value with (c1), (c2), and (c3). (i) of Fig. 5 shows the distribution of \(\alpha_i\) and \(\beta_i\) with conditions of (8-10) indicated by \(O\)'s and (c1), (c2') and (c3) represented by \(X\)'s. Similarly, \(\alpha_j\) and \(\beta_j\) and \(\alpha_i\) and \(\beta_i\) are shown in (ii) and (iii), respectively.

**Figure 4.** The coefficients of non-maximal states that approximate the weaker link are represented by \(\alpha_i\)’s (top) and \(\beta_i\)’s (bottom).

**Figure 5.** A comparison between the coefficient (indicated by \(O\)'s) that satisfies (c1),(c2),(c3), that yields the weaker link, and the coefficient (indicated by \(X\)'s) with (c1),(c2') and (c3) that approximates the optimal result is shown. The coefficients that approximate the weaker link are just as widely disbursed as the coefficients of optimal outcome.
Remarks

Ever since the development of quantum theory initiated a century ago, its precise meaning has been debated (Peres, 1997). However, in the past few decades, this philosophical argument turned into a physically realizable and useful technology. At the heart of the recent development in quantum information science sits entanglement. In this paper, entanglement swapping schemes involving two 3-level non-maximal states have been examined using numerical methods. This study has shown a different class of coefficients that yields an outcome close to the weaker link. This result may be useful in creating long distance maximal entanglement, a key issue in several recent information technologies based on quantum theory.

References


