Domination Polynomial of Cartesian Product of Some Graphs

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ABSTRACT

In this paper, the domination polynomial of the Cartesian product of the paths and the cycles has been discussed.

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1 Introduction

Let G=(V,E) be a simple graph with vertex set V and edge set E. A set D⊆ V is a dominating set of G, if every vertex in V-D is adjacent to atleast one vertex in D. The domination number is the minimum cardinality of a dominating set in G. For detailed treatment of dominations, one can refer [1]. The domination polynomials of product of graphs is an interesting area of research in graph theory. Saeid Alikhani, and Yee-Hock Peng [2] have studied dominating sets and domination polynomials of cycles. Tomer Kotek, James Preen, Peter Tittmann[3], studied domination polynomials of graph products. In particular, they studied the domination polynomials of product of complete graphs with complete graphs, and complete graphs with paths. Abdul Jalil M.Khalaf and Sahib Shayyal Kahat[4] studied dominating sets and domination of complete graphs with missing edges, A.Vijiayan and K.Lal Gipson[5] studied domination polynomials of square of cycles. As in any branch of graph theory, there are many open problems which deserve attention in the area of domination polynomials. In this paper, we present an approach to study the domination polynomial of the cartesian product of paths and cycles, and path(P_2) and complete graphs.

2 Pre-Requisties

A path is a simple graph with the vertex set $V = \{x_1, x_2, \dots x_n\}$, and edge set $E = \{x_1 x_2, x_2 x_3, \dots x_{n-1} x_n\}$. A path with n vertices is denoted as P_n. Similarly, a cycle is a simple graph with vertex set $V = \{x_1, x_2, \dots x_n\}$ and edge set $E = \{x_1 x_2, x_2 x_3, \dots x_{n-1} x_n, x_n x_1\}$. A cycle with n vertices is denoted as C_n. The cartesian product of two graphs G and H is a graph denoted by G H, vertex set $V(G)\times V(H)$ and two vertices (u, v) and (u^0, v^0) are adjacent if and only if $u = u^0$ and $vv^0 \in E(H)$ or $v = v^0$ and $uu^0 \in E(G)$. In this paper, we consider the cartesian product of paths and cycles.

2.1 Definition

The domination polynomial D(G, x) is defined as $D(G, x) = \sum_{i=1}^{|v|} d_i(G)x^i$

Where $d_i(G)$ is the number of dominating sets of size $'i'$ in G.

2.2 Example

Consider the graph given in Fig 2.2

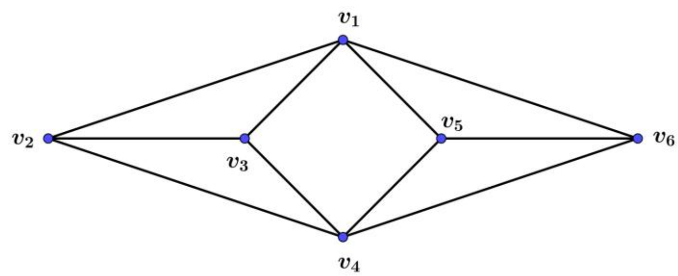

Figure 2.2: A simple graph

It can be found that the number of dominating sets of the above graph of sizes 1,2,3,4,5,6 are respectively 0,13,10,9,3,1. Thus, the domination polynomial of the graph given in fig 2.1 is

$$D(G,x) = 0x^1 + 13x^2 + 10x^3 + 6x^4 + 3x^5 + 1x^6$$

Saeid Alikhani and Yee-Hock Peng[2] have studied the domination poly- nomial of paths graphs and obtained

$D(p_{n+1},\lambda) = x[D(p_n,\lambda)+D(p_{n-1},\lambda)+D(p_{n-2},\lambda)]$ Where $D(p_0, x) = 1$; $D(p_1, x) = x$; $D(p_2, x) = x^2 + 2x$ They also studied domination polynomial of cycles.

Abdul Jalil M.Khalaf and Sahib Shayayal Kahat[4] have studied the domi- nation polynomial of complete graphs and obtained the recurrence formula $D(k_n, x) = D(k_{n-1}, x) + xD(k_{n-1}, x) + x$ for $n \geq 3$.

In the year 2013, Tomer Kotek and others [3] have studied domination polynomial of graph products. In particular, they studied the domination polynomial of products with complete graphs. In this paper we study the domination polynomial of cartesian product of paths and cycles.

3 Domination polynomial of cartesian product of cycles and paths

Domination is an important graph theoretic concept. It provides a represent- tation set of vertices in a graph G instead of whole set of vertices to extend various vertices related concepts in graph theory. Whenever mathematics is involved, it is an usual phenomenon to discuss the trivial cases. It is a simple observation that if a graph has n vertices and there is a vertex of degree $n - 1$ then the singleton set consisting only that vertex is a dominating set. Similarly, if a set D containing r vertices in cycle C_n, $r < n$; is a dominating set then any set D' containing r+1 vertices in C_n is also a dominating set.

For any graph G, it can be seen that $d_r(G) \leq nC_r$. From the above observations, we have the following lemma

3.1 Lemma

For any cycle C_n, i) $d_{n-1}(C_n) = nC_{n-1}$

ii) $d_{n-2}(C_n) = nC_{n-2}$

iii) $d_n(C_n) = nC_n = n$

iv) $d_{n-3}(C_n) < nC_{n-3}$.

In this paper, we discuss the dominating polynomial of cartesian product of paths and cycles. First we consider the cartesian product $P_2\ C_3$.

$P_2\ C_3 = \{w_{ij} = (u_i, v_j)/i = 1,2,\ j = 1,2,3;\ w_{ij} \sim w_{kl}$ if $i = k$ or $j = l\}$.

The vertex w_{ij} is said to be contributed by u_i and v_j.

The graph of $P_2\ C_3$ is given in the fig 3.1

We know that $\deg(w_{ij}) = 3$, and $|V(P_2\ C_3)| = 6$. Hence , there will not be any singleton set as a dominating set for $P_2 C_3$, because if $\{w_{ij}\}$ is a dominating set in $P_2 C_3$, there $\deg(w_{ij})$ must be 5, this is not true. Hence d_1

$(P_2 C_3) = 0$.

Now the vertex $w_{ij} \sim w_{ik}$, and also the $w_{ij} \sim w_{lj}$, and also the number of vertices which differ in both subscripts of w_{ij} is only two. For example, the vertices which differ in both subscripts of w_{11} are w_{22} and w_{23}, and they are adjacent to each other. Hence for every w_{ij} , the number of non-adjacent vertices is 2. Therefore, the vertex w_{ij} together with any one of such non-adjacent vertex form a dominating set. The number of dominating sets obtained in this way is 2. This is possible for each w_{ij} for fixed i. Hence total number of dominating sets obtained in this way is 6. Also for every vertex w_{ij}, the set.

$\{(w_{ij},\ w_{i+1,j})/i = 1,\ j = 1, 2, 3\}$ is also a dominating set. The number of such dominating sets is 3. Hence, there are 9 dominating sets of size 2 in $P_2 C_3$. Hence $d_2(P_2 C_3) = 9$. The table 3.1 gives all the nine dominating sets.

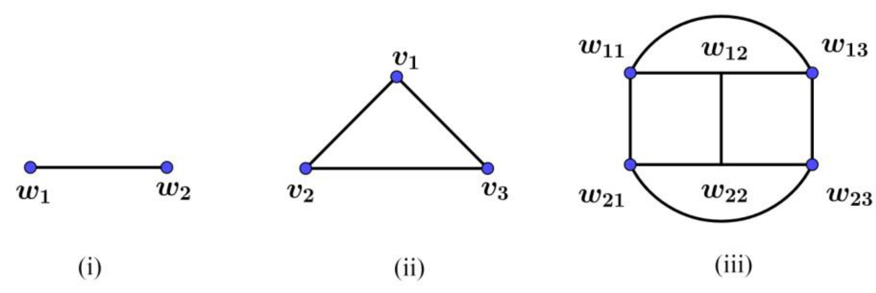

Figure 3.1: (i) Path P_2 (ii) Cycle C_3 (iii) Cartesian product $P_2\ C_3$

S.No	Dominating	Distance
1	w11 , w22	2
2	w11 , w23	2
3	w12 , w21	2
4	w12 , w23	2
5	w13 , w21	2

Table 3.1 Minimum dominating sets of size 2 in P_2 C_3.

It can be observed that is (w_{ij} ,w_{lk}) form a dominating set if i= l. i.e., The dominating set is obtained when w's are contributed by one vertex from path, and one vertex from cycle, or both the vertices from the path. Therefore, if we consider any three vertices w_{ij} , then there should be atleast one vertex is contributed from path, and one vertex is contributed from cycle. Hence any three vertices form a dominating set.

Therefore $d_3(P_2 C_3) = 6C_3$ and hence

$d_4(P_2 C_3) = 6C_4$, $d_5(P_2 C_3) = 6C_5$, $d_6(P_2 C_3) = 6C_6$.

Hence $D(P_2 C_3, x) = 0x^1 + 9x^2 + 6C_3 x^3 + 6C_4 x^4 + 6C_5 x^5 + 6C_6 x^6$.

$= 0x^1 + 9x^2 + 20x^3 + 15x^4 + 6x^5 + 1x^6$.

As an another illustration consider the graph $P_2 C_4$ given in Fig 3.2 (i) In the distance based structure given in Fig 3.2(ii), the vertex at level 0 is a root, and vertex at last level is top of the structure. It can be easily observed that no single vertex form a dominating set. Hence $d_1(P_2 C_4)$ =0. The following lemma gives the other coefficients.

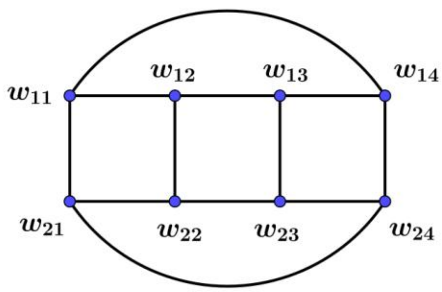

Figure 3.2: (i) The graph P_2 C_4

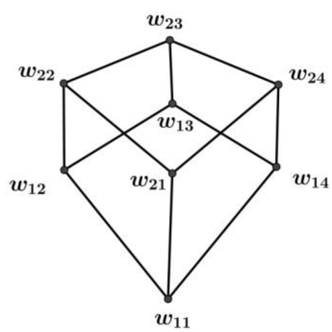

Figure 3.2: (ii) Distance based structure of P_2 C_4

3.2 Lemma

$d_2(P_2 C_4) = 4$; $d_3(P_2 C_4) = 24$

Proof

It can be easily found that only the vertices at root and top will form a dominating set in $P_2 C_4$. There are four such combinations in the set of 8 vertices. Therefore, there are 4 dominating sets of size 2. Hence $d_2(P_2 C_4)$ =

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4. Any dominating set of two elements can also be made as a dominating set of three elements, four elements, five elements and so on.... the following is a list of 2-vertices that form a dominating set.

i) w_{11}, w_{23}

ii) w_{12}, w_{24}

iii) w_{13}, w_{21}

iv) w_{14}, w_{22}

Inclusion of any one vertex in any one of the lists give us a dominating set of size 3. For each list there are 6 possible vertices. Since the vertices in each list are distinct, the all possible collection of vertices form distinct sets. The number of such dominating sets is 24. Now we claim that no other dominating set of size 3 are there in $P_2 C_4$. Note that the 4 dominating sets of size 2 are isolated dominating sets. Hence, the dominating partner of a vertex in each dominating set is unique. Now, let us consider the non-dominating set of size 2 in $P_2 C_4$. The number of non-dominating partners of vertices belonging to those non-dominating sets is 4. For example, the non-dominating partners of the vertices in the non-dominating set $\{w_{11}, w_{12}\}$ are $w_{13}, w_{14}, w_{21}, w_{22}$. None of these vertices, will give as a dominating set of size 3 in $P_2 C_4$. Hence $d_3(P_2 C_4) = 24$.

3.3 Observation

The graph obtained replacing top to root or root to top will give the same structure providing the same dominating sets.

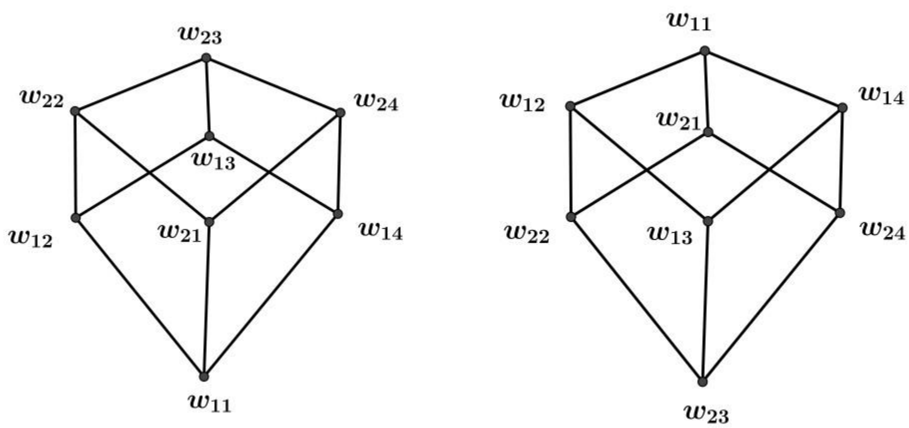

Figure 3.3: Different structures of P_2 C_4

The set of four elements which does not form the dominating set in P_2 C_4 given in the above diagram are

i) $w_{11}, w_{12}, w_{21}, w_{14}$

ii) $w_{23}, w_{22}, w_{13}, w_{24}$

iii) $w_{11}, w_{14}, w_{13}, w_{24}$

iv) $w_{11}, w_{21}, w_{22}, w_{24}$

v) $w_{11}, w_{12}, w_{22}, w_{13}$

vi) $w_{23}, w_{22}, w_{12}, w_{21}$

vii) $w_{23}, w_{13}, w_{12}, w_{14}$

viii) $w_{23}, w_{24}, w_{21}, w_{14}$

From the above observations, we get

3.4 Lemma

There are 4 distance based structure for $P_2 C_4$, with different pairs of root and top.

If D is a dominating set in a distance based structure of $P_2 C_4$ with root u and top v, if and only if D is also a dominating set in distance based structure of $P_2 C_4$, with root v and top u. Hence, we get the following lemma.

3.5 Lemma

$$d_4(P_2 C_4) = 38$$

3.6 Lemma

There are eight 4-set of vertices in the distance based structure of P_2 C_4 which are not dominating sets

3.7 Theorem

The dominating polynomial of $P_2 C_4$ is

$$D(P_2 C_4, X) = 0X^1 + 4X^2 + 24X^3 + 38X^4 + 56X^5 + 28X^6 + 8X^7 + X^8$$

Now, let us discuss the number of isolated dominating sets in a graph. For this let us take the graph P_3 C_3, given in Fig 3.7 No three vertices w_{ij}, j = 1,2,3 for fixed 'i' form a dominating set. Hence the coefficient of x^3 does not exceed $9C_3 - 3$, Also , for any two vertices w_{ij}, for fixed i, there is only one vertex that gives a dominating set together with that two vertices. Such sets are isolated dominating sets.

3.8 Lemma

There are $2(3C_2)$ isolated dominating sets in $P_3 C_3$. Before proving this lemma, let us define isolated dominating set precisely.

3.9 Definition

Let D be a dominating set of size n. A subset D^0 of D with cardinality n-1 is the dominantly restricted subset of D , if D is unique containing D^0.

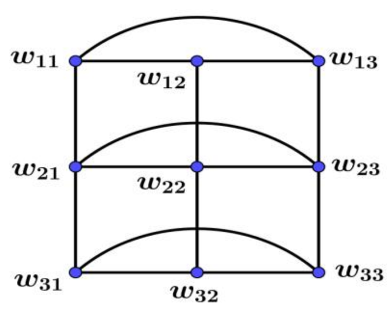

Figure 3.7: P_3 C_3

3.10 Example

The subset $\{w_{11}, w_{12}\}$ of $V(P_3 C_3)$ is a dominantly restricted subset of the dominating set $\{w_{11}, w_{12}, w_{33}\}$, whereas $\{w_{11}, w_{33}\}$ is not a dominantly re- stricted subset of $\{w_{11}, w_{12}, w_{33}\}$ as there are dominating sets $\{w_{11}, w_{21}, w_{33}\}$, $\{w_{11}, w_{22}, w_{33}$

}, {w_{11}, w_{23}, w_{33}} containing {w_{11}, w_{33}}. A dominating set D is an isolated dominating set if it contains dominantly restricted subset D^0. Now let us prove the lemma.

3.8 Proof of Lemma

It can be seen that the two vertices w_{ij}, for fixed i = 1 or 3, are dominantly restricted subsets of the dominating set containing these two vertices together with w_{kl}, where k = 1 if i = 3 or k = 3 if i = 1, and i = j. The complete list of such dominating sets are given below. 58

i) {w_{11}, w_{12}, w_{33}} ii) {w_{11}, w_{13}, w_{32}} iii) {w_{12}, w_{13}, w_{31}} iv) {w_{31}, w_{32}, w_{13}} v) {w_{31}, w_{33}, w_{12}} vi) {w_{32}, w_{33}, w_{11}}

The number of such sets are $2(3C_2) = 6$. Hence the lemma. Also any three vertices w_{ij} i = 1,2,3, for fixed j is a dominating set. Such a dominating set is a non-isolated dominating set. There are 3 non- isolated dominating sets for every pair of w_{ij}, for fixed j. Therefore, the number of such non-isolated dominating sets of order 3 is $3(3C_2)x3 = 9(3C_2)$, when the vertices are of the form w_{ij}, for fixed j. The complete list of the isolated dominating sets, and non-isolated dominating sets of order 3 in P_3 C_3 is given in table 3.1.

List:1	List:2
Isolated dominating sets	Non-isolated dominating sets
w_{11}, w_{12}, w_{33}	w_{11}, w_{21}, w_{31} w_{11}, w_{13}, w_{32} $w_{11}, w_{21},$
w_{32} w_{12}, w_{13}, w_{31}	w_{11}, w_{21}, w_{33} w_{31}, w_{32}, w_{13} $w_{11},$
w_{31}, w_{21} w_{31}, w_{33}, w_{12}	w_{11}, w_{31}, w_{22} w_{32}, w_{33}, w_{11}
	w_{11}, w_{31}, w_{23} w_{21}, w_{31}, w_{11}
	w_{21}, w_{31}, w_{12}
	w_{21}, w_{31}, w_{13}

Table 3.1 Isolated and Non-isolated dominating sets in P_2 C_3. There are 9 dominating sets are constructed by taking 2 vertices from w_{11}, w_{21}, w_{31} and third vertex from the other Row not involved in any of 2 Rows from the 3 vertices mentioned above. These 9 sets are considered with respect to first Column. Similarly , other 9 sets can also be formed with respect to Column 2, Column 3, and Row 2. Thus there are 4x9=36 non-isolated dominating set in P_3 C_3. In total, there are 42(isolated-6, non-isolated-36) dominating sets of order 3. These 42 sets are collected only by collecting two vertices in a Row or in a Column. All 3 vertices in a Row cannot form a dominating set, but all 3 vertices in a Column form a dominating set. Now they only case left over in identifying dominating set of order 3 in P_2 C_3 is the selection of one vertex in each Column and each Row. There are 6 such sets. Hence the total number of dominating sets of size 3 in P_3 C_3 is 48.

From the above discussion we observe the following.

i) Any two vertices in a Row (except the second Row) contribute an iso- lated dominating set. (There are 6 such isolated dominating sets).

ii) Any two vertices in the second row contribute 3 non-isolated dominating sets.

iii) One vertex in each row contribute two non-isolated dominating sets other than the dominating set obtained in (ii) (There are 6 such dominating sets)

iv) All the vertices in a row do not form a dominating set.

v) All 3 vertices in a column form a dominating set (There are 3 such sets)

vi) Any two vertices in a column contribute 3 non-isolated dominating sets. Hence there are 3 × 3C2 × 3 (Each pair has 3 dominating sets, and there are 3C2 pairs in a column, and $P_3 C_3$ has 3 columns). These include the dominating sets obtained in (v).

vii) One in each column contribute a dominating set obtained either in (ii), or (iii), or (vi).

viii) The total number of dominating sets of size 3 in P_3 C_3 is 6+3 ×

3C2 +6+3×3C2×3 = 48.

4 Domination polynomial of cartesian product of P_2 and K_n

In this section, we discuss the domination polynomial of P_2 and the complete graph K_n. The graphs $P_2 K_3$ and $P_2 K_4$ are given in Fig 4.1

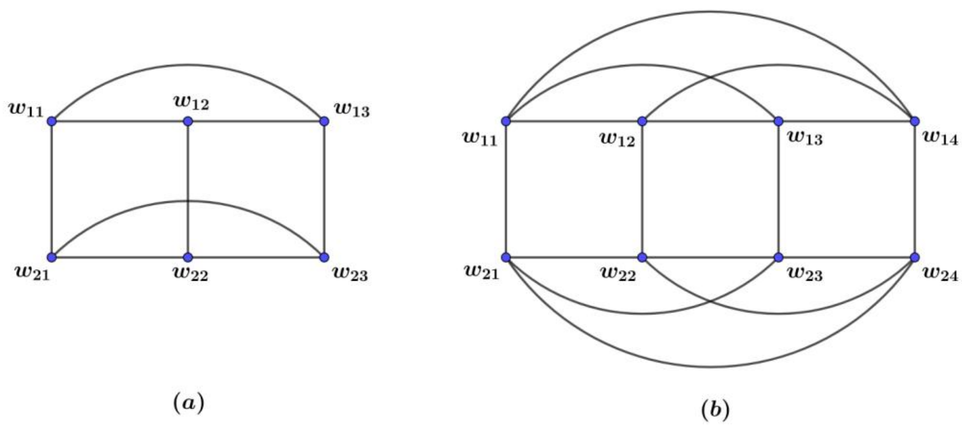

Figure 4.1: (a) P_2 K_3 (b) P_2 K_4

It can be found that any two vertices contributed by same vertex of P_2 cannot form a dominating set. Hence, any subset consisting of two vertices except the set of vertices contributed by the same vertex of the path is a dominating set. i.e, we have

$D(P_2 K_3) = [6C_2 - 2(3C_2)]x^2 + 6C_3 x^3 + \ldots..$

$D(P_2 K_4) = [8C_2 - 2(4C_2)]x^2 + [8C_3 - 2(4C_3)]x^3\ 8C_4 x^4 + \ldots..$

$D(P_2 K_5) = [10C_2 - 2(5C_2)]x^2 + [10C_3 - 2(5C_3)]x^3 + [10C_4 - 2(5C_4)x^4 +$

$10C_5 x^5 + \ldots 10C_{10} x^{10}$. and so on., On generalizing we get the following theorem.

Theorem 4.1

$(P_2 K_n) = [2nC_2 - 2(nC_2)]x^2 + [2nC_3 - 2(nC_3)]x^3 + \ldots.[$ $-$

$2nC_n$

$n-1$ $)x$

$+2nC_n x^n + 2nC_{n+1} x^{n+1} + 2nC_{n+2} x^{n+2} + \ldots 2nC_{2n}$

n

x^{2n}

-1

$1 - 2(nC$

$$= \sum_{r=2}^{n-1} [2nC_r - 2(nC_r)]x^r + \sum_{s=n}^{2n} 2nC_s x^s$$

5 Conclusion

In this paper, the domination polynomial of cartesian product of some special graphs of particular types have been discussed.

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