Generalized Pascal's Triangle and its Properties

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Abstract

Pascal Triangle named after French mathematician and Scientist Blaise Pascal contain rich source of mathematical properties. Ever since, Pascal provided its mathematical significance, several mathematicians and scientists had shown great interest in discovering new properties that can exist among numbers in Pascal's Triangle. In this paper, we had generalized the Pascal's Triangle in a novel way and had derived some interesting properties related to the generalized forms.

Key Words: Pascal's Triangle, Pascal's Triangle of mth order, Binomial Coefficients, South – East Diagonal, Hockey Stick Property.

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Introduction

Though the mathematical properties of Pascal's triangle was first described by Blaise Pascal in relation to describing probability distributions of random events, mathematicians from ancient India and China knew about this triangle and had mentioned it explicitly several centuries before Pascal. Nevertheless, the importance and utility of Pascal's Triangle was first provided by Pascal and hence it was named in his honor. In this paper, we try to generalize the Pascal's Triangle and derive some interesting results regarding the generalized Pascal triangle structures.

Definitions

1. If a, b are two real numbers and if n is a whole number, then $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ (2.1)

The quantities $\binom{n}{k}$ for $k = 0,1,2,3,\ldots,n$ which occur as coefficients in the binomial expansion as in (2.1), are called binomial coefficients.

We can view binomial coefficients as number of ways of selecting k objects out of n distinguishable objects. These numbers occur in plenty of counting problems in mathematics and were considered to be one of the significant families of numbers in combinatorics. We now list some of the well known combinatorial identities related to binomial coefficients.

2. Properties of Binomial Coefficients

2.1. $\binom{n}{r} = \binom{n}{n-r}$ (2.2)

2.2. $\binom{n+1}{r+1} = \binom{n}{r+1} + \binom{n}{r}$, $0 \le r \le n$ (2.3)

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2.3. $\sum_{r=0}^{n}\binom{n}{r}=2^n \quad (2.4)$

2.4. $\sum_{k=r}^{n}\binom{k}{r}=\sum_{k=r}^{n}\binom{k}{k-r}=\binom{n+1}{r+1} \quad (2.5)$

(Hockey Stick Identity)

Pascal's Triangle

Pascal's Triangle is collection of numbers in triangular array whose entries are binomial coefficients. We designate the row numbers of the triangle by n, where n = 0, 1, 2, 3,... The rth element in nth row of the Pascal's triangle is given by $T_1(n,r)=\binom{n}{r}$, $0\le r\le n$ (3.1) where we assume that $\binom{0}{0}=1$. With the entries as defined in (3.1), the Pascal's triangle is displayed in Figure 1. We call Pascal's Triangle shown below as Pascal's Triangle of 1st order.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
⋰ ⋮ ⋱

Figure 1. Pascal's Triangle of 1st order

4. 2nd Order Pascal's Triangle

The rth element in nth row entry of Pascal's triangle of 2nd order is defined as $T_2(n,r)=\binom{n}{r}+\binom{n-1}{r-1}$, $1\le r\le n$ (4.1) where n = 0, 1, 2, 3,... and $T_2(n,0)=1$. With this notion, we see that $T_2(n,n)=\binom{n}{n}+\binom{n-1}{n-1}=2$.

The entries of Pascal's triangle of 2nd order are given by

1
1 2
1 3 2
1 4 5 2
1 5 9 7 2
1 6 14 16 9 2
1 7 20 30 25 11 2
1 8 27 50 55 36 13 2
1 9 35 77 105 91 49 15 2
1 10 44 112 182 196 140 64 17 2
. . . .

Figure 2. Pascal's Triangle of 2nd order

We observe the entries along South-East diagonals in Figure 2. The first South-East diagonal entries are all 2 except the first entry which is 1. The second, third, fourth South-East diagonal entries are odd numbers, square numbers and square pyramidal numbers respectively. We now prove this formally through following theorems.

1. Theorem 1

The entries in second South – East diagonal in Pascal's Triangle of 2nd order are precisely the set of all odd natural numbers. That is, $T_2(n,n-1)=2n-1$ (4.2), $n=1,2,3,4,\ldots$

Proof: From (2.2) and (4.1), we have

$$T_2(n,n-1)=\binom{n}{n-1}+\binom{n-1}{n-2}=\binom{n}{1}+\binom{n-1}{1}=n+(n-1)=2n-1$$

as required. In fact, these entries were highlighted in red color in Figure 2.

2. Theorem 2

The entries in third South – East diagonal in Pascal's Triangle of 2nd order are precisely the set of all square numbers. That is, $T_2(n,n-2)=(n-1)^2$ (4.3), $n=2,3,4,5,\ldots$

Proof: From (2.2) and (4.1), we have

$$T_2(n,n-2)=\binom{n}{n-2}+\binom{n-1}{n-3}=\binom{n}{2}+\binom{n-1}{2}=\frac{n(n-1)}{2}+\frac{(n-1)(n-2)}{2}=(n-1)^2$$

as required. These entries were highlighted in blue color in Figure 2.

3. Theorem 3

The entries in fourth South – East diagonal in Pascal's Triangle of 2nd order are precisely the set of all square pyramidal numbers. That is,

$$T_2(n,n-3)=1^2+2^2+\cdots+(n-2)^2=\frac{(n-2)(n-1)(2n-3)}{6} \quad (4.4),\ n=3,4,5,6,\ldots$$

Proof: From (2.2) and (4.1), we have

$$T_2(n,n-3)=\binom{n}{n-3}+\binom{n-1}{n-4}=\binom{n}{3}+\binom{n-1}{3}=\frac{n(n-1)(n-2)}{6}+\frac{(n-1)(n-2)(n-3)}{6}$$
$$=\frac{(n-1)(n-2)}{6}[n+n-3]=\frac{(n-2)(n-1)(2n-3)}{6}=1^2+2^2+\cdots+(n-2)^2$$

Since sum of consecutive squares are square pyramidal numbers, it follows that $T_2(n,n-3)$ represent set of all square pyramidal numbers for n = 3, 4, 5, 6,... These entries were highlighted in pink color in Figure 2.

4. Theorem 4

Except first row, the sum of all entries in the nth row of Pascal's triangle of 2nd order is $3\times 2^{n-1}$ (4.5), n = 1,2,3,4,...

Proof: Using (2.4) and (4.1), we have

$$\sum_{r=0}^{n}T_2(n,r)=\sum_{r=0}^{n}\left[\binom{n}{r}+\binom{n-1}{r-1}\right]=\sum_{r=0}^{n}\binom{n}{r}+\sum_{r=1}^{n}\binom{n-1}{r-1}=2^n+2^{n-1}=3\times 2^{n-1}$$

as required.

5. Theorem 5 (Hockey Stick Identity for Pascal's Triangle of 2nd order)

$$\sum_{k=r}^{n}T_2(k,k-r)=T_2(n+1,n-r) \quad (4.6)$$

Proof: Using (2.5) and (4.1), we have

$$\sum_{k=r}^{n}T_2(k,k-r)=\sum_{k=r}^{n}\left[\binom{k}{k-r}+\binom{k-1}{k-r-1}\right]=\sum_{k=r}^{n}\binom{k}{k-r}+\sum_{k=r}^{n}\binom{k-1}{k-r-1}$$
$$=\sum_{k=r}^{n}\binom{k}{r}+\sum_{k=r}^{n}\binom{k-1}{r}=\binom{n+1}{r+1}+\binom{n}{r+1}=\binom{n+1}{n-r}+\binom{n}{n-r-1}$$
$$=T_2(n+1,n-r)$$

This completes the proof.

Higher Order Pascal's Triangle

The rth element in nth row entry of Pascal's triangle of mth order, where m is a positive integer, is defined as $T_m(n,r)=\binom{n}{r}+(m-1)\binom{n-1}{r-1},\ 1\le r\le n$ (5.1), n = 0, 1, 2, 3, . . . and $T_m(n,0)=1$. With this notion, we see that

$$T_m(n,n)=\binom{n}{n}+(m-1)\binom{n-1}{n-1}=1+(m-1)=m$$

The entries of Pascal's triangle of mth order are given by

1
$1 \quad m$
$1 \quad m+1 \quad m$
$1 \quad m+2 \quad 2m+1 \quad m$
$1 \quad m+3 \quad 3m+3 \quad 3m+1 \quad m$
$1 \quad m+4 \quad 4m+6 \quad 6m+4 \quad 4m+1 \quad m$
$1 \quad m+5 \quad 5m+10 \quad 10m+10 \quad 10m+5 \quad 5m+1 \quad m$
$1 \quad m+6 \quad 6m+15 \quad 15m+20 \quad 20m+15 \quad 15m+6 \quad 6m+1 \quad m$
$1 \quad m+7 \quad 7m+21 \quad 21m+35 \quad 35m+35 \quad 35m+21 \quad 21m+7 \quad 7m+1 \quad m$
. . . .

Figure 3. Pascal's Triangle of mth order

We note that if m = 1, then the triangle in Figure 3 coincide with that of in Figure 1. In this sense, the usual Pascal Triangle displayed in Figure 1 becomes the special case of Pascal Triangle of mth order shown in Figure 3.

1. Theorem 6

Except first row, the sum of all entries in the nth row of Pascal's triangle of mth order is $2^{n-1}\times(m+1)$ (5.2), n = 1,2,3,4,...

Proof: Using (2.4) and (5.1), we have

$$\sum_{r=0}^{n}T_m(n,r)=\sum_{r=0}^{n}\left[\binom{n}{r}+(m-1)\binom{n-1}{r-1}\right]=\sum_{r=0}^{n}\binom{n}{r}+(m-1)\sum_{r=1}^{n}\binom{n-1}{r-1}$$
$$=2^n+(m-1)2^{n-1}=2^{n-1}\times(m+1)$$

This completes the proof.

2. Theorem 7 (Hockey Stick Identity for Pascal's Triangle of mth order)

$$\sum_{k=r}^{n}T_m(k,k-r)=T_m(n+1,n-r) \quad (5.3)$$

Proof: Using (2.5) and (5.1), we have

$$\sum_{k=r}^{n}T_m(k,k-r)=\sum_{k=r}^{n}\left[\binom{k}{k-r}+(m-1)\binom{k-1}{k-r-1}\right]$$
$$=\sum_{k=r}^{n}\binom{k}{r}+(m-1)\sum_{k=r}^{n}\binom{k-1}{r}$$
$$=\binom{n+1}{r+1}+(m-1)\binom{n}{r+1}=\binom{n+1}{n-r}+(m-1)\binom{n}{n-r-1}$$
$$=T_m(n+1,n-r)$$

This completes the proof.

Conclusion

In this paper, we had generalized the usual Pascal's triangle to define Pascal's triangle of mth order as defined in (5.1) and displayed in Figure 3. We notice that for m = 1, the generalized Pascal's triangle reduces to usual Pascal's triangle. Thus the

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well known Pascal's triangle is a special case for m = 1. In other words, the first order Pascal triangle defined in (5.1) is the usual Pascal's triangle.

In section 4, introducing Pascal's triangle of 2nd order, displayed in Figure 2, we had proved that the entries in the second, third, fourth South – East diagonals are odd numbers, square numbers and square pyramidal numbers respectively. These theorems provide ways of identifying special class of numbers upon generalizing the usual Pascal's triangle.

We notice from the definition of Pascal's triangle of 2nd and mth order that the entries are not symmetrical with respect to the symmetrical line as we get in usual Pascal's triangle, nevertheless the famous Hockey Stick Property holds true for our generalized Pascal's triangle, the proof of which can be seen in theorems 5 and 7.

In theorem 6, we had obtained a nice expression for finding sum of entries in nth row of Pascal's triangle of mth order which turns out to be $2^{n-1} \times (m+1)$. We notice that for m = 1, the sum of row entries is 2^n which is known through usual Pascal's triangle. Moreover, if we choose m to be a Mersenne number of the form $m = 2^t - 1$ for some natural number t, then the row sum in generalized Pascal's triangle of mth order would be powers of 2.

Thus, by generalizing the Pascal triangle to that of mth order, we had obtained more general results in this paper. The concepts discussed in this paper provide more opportunities to create several possible generalizations of the usual Pascal's triangle.

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